ADJUSTMENT OF CABLE - STAYED BRIDGES

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ABSTRACT

The adjustment of cable - stayed bridges needs the shortening or lengthening of several or all cables. Changing the length of a single cable produces the change in forces at other stays and at the stiffening girder. The calculation of the forces due to the adjustment of one stay seems to be advantageous by the force method. The paper gives an analytical way to the calculation of a bridge with a given arrangement. The chosen primary system leads to a system of equations having a pentadiagonal coefficient matrix. Because of the complexity of the task, even in the case of the regular form of the bridge, the solution is given by a recursion.

1. INTRODUCTION

It is generally known that the accuracy of assembly of cable-stayed bridges cannot be perfect. An adjustment of cables is needed in great majority of cases [5]. During this procedure, in general, the stays are stressed or released according to the deviation of the force acting in them to the designed value. A cable-stayed bridge is a statically indeterminate structure to at least as high degree as the number of the stays. Obviously, the change of the length of a stay indicates forces in all other ones, as well as in the stiffening girder.

As it was previously shown [8], an explicit solution is not precluded even in the case of regular arrangement of the structure. The problem of adjustment could be treated by the deformation method. In this case, the inverse of the coefficient matrix of the system should be multiplied by a load vector having a single element (a force) which is not equal to zero. The application of the force method for this task is more comfortable. Namely, the *influence coefficient* [4] gives – even if not quite directly – the effect of elongation or shortening of a stay. N.B.: From this point of view, a primary system defined by cutting the stays would be more convenient, but less advantageous considering the procedure and accuracy of the calculation. The force method also enables us to reckon with the incidental yielding of a stay or occurring of a plastic hinge in the stiffening girder [3], [7]. Speaking about the forces acting in a cable-stayed bridge, it should be mentioned that the situation is similar to the case of branchy systems [2], the signs are but opposite.

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The problem will be discussed for a single-bay bridge, the stays start from the top of the tower and are tied centrally to the stiffening girder at equidistant points.

2. THE STRUCTURAL ARRANGEMENT

The arrangement of the cable stayed structure is shown in Fig. 1. The geometric characteristics and the stiffness symbols are also given. It is supposed that the tower is absolutely rigid, the axial and the normal deformations of the stiffening girder are neglected and its flexural stiffness is constant, the cross section of the stays is uniform, as well as the distance between their centric joining points to the girder.

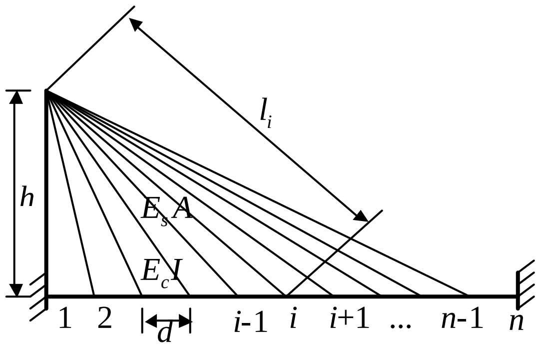

Fig. 1. The arragement of the structure

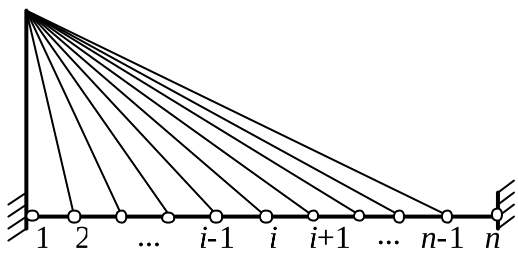

Fig. 2. The primary system

3. THE PRIMARY SYSTEM AND THE SYSTEM OF EQUATIONS

First order theory will be considered. The primary system of the force method can be seen in Fig. 2. It is formed by applying hinges at both constrained ends and at the joining points of cables along the girder.

The unknowns are the moments of the stiffening girder at these points. The elements of the coefficient matrix of the system are rotations round the hinges due to the unit moment-pairs applied there.

The system of equations can be written in the form

$$\mathbf{A}\,\mathbf{x} = \mathbf{a_0}\,,$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the vector of the unknown moments and $\mathbf{a_0}$ is the load vector. The shape of the matrix $\mathbf{A}$ is penta-diagonal:

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & & & & & & & & & \\
a_{21} & a_{22} & a_{23} & a_{24} & & & & & & & & \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & \\
 & & a_{i-1,i-3} & a_{i-1,i-2} & a_{i-1,i-1} & a_{i-1,i} & a_{i-1,i+1} & & & & & \\
 & & & a_{i,i-2} & a_{i,i-1} & a_{i,i} & a_{i,i+1} & a_{i,i+2} & & & & \\
 & & & & a_{i+1,i-1} & a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} & a_{i+1,i+3} & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & & a_{n-2,n-4} & a_{n-2,n-3} & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\
 & & & & & & & & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
 & & & & & & & & & a_{n,n-2} & a_{n,n-1} & a_{n,n}
\end{bmatrix}
$$

The elements of the symmetric coefficient matrix are of the primary system:

$$a_{11} = \frac{1}{3}\frac{d}{E_c I} + \frac{1}{E_s A d^2 h^2} l_2^3,$$

$$a_{12} = \frac{1}{6}\frac{d}{E_c I} - \frac{2}{E_s A d^2 h^2} l_2^3,$$

$$a_{13} = \frac{1}{E_s A d^2 h^2} l_2^3,$$

$$a_{22} = \frac{2}{3}\frac{d}{E_c I} + \frac{1}{E_s A d^2 h^2} \left\{4 l_2^3 + l_3^3\right\},$$

$$a_{ii} = \frac{2}{3}\frac{d}{E_c I} + \frac{1}{E_s A d^2 h^2} \left\{l_{i-1}^3 + 4 l_i^3 + l_{i+1}^3\right\},$$

$$a_{i,i+1} = \frac{1}{6}\frac{d}{E_c I} - \frac{2}{E_s A d^2 h^2} \left\{l_i^3 + l_{i+1}^3\right\},$$

$$a_{i,i+2} = \frac{1}{E_s A d^2 h^2} l_{i+1}^3,$$

$$a_{n-1,n-1} = \frac{2}{3}\frac{d}{E_c I} + \frac{1}{E_s A d^2 h^2} \left\{l_{n-2}^3 + 4 l_{n-1}^3\right\},$$

$$a_{n,n-1} = \frac{1}{6}\frac{d}{E_c I} - \frac{2}{E_s A d^2 h^2} l_{n-1}^3,$$

$$a_{n,n-2} = \frac{1}{E_s A d^2 h^2} l_{n-1}^3,$$

$$a_{nn} = \frac{1}{3}\frac{d}{E_c I} + \frac{1}{E_s A d^2 h^2} l_{n-1}^3,$$

where $l_i = \left\{h^2 + (i-1)^2 d^2\right\}^{\frac{1}{2}}$.

4. THE INVERSE OF THE COEFFICIENT MATRIX

In order to perform the calculations, it is advisable to partition the coefficient matrix into second order blocks. For this purpose, let us assume that n is an even number, i.e.

$$n = 2m.$$

Introducing

$$\mathbf{Z}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix},\ \mathbf{Z} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\ \mathbf{Z}_m = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix},\ \mathbf{V} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

the coefficient matrix can be written in the form

$$E_s A d^2 h^2 \mathbf{A} = k \begin{bmatrix} \mathbf{Z}_0 & \mathbf{V} & & & & \\ \mathbf{V}^\mathbf{T} & \mathbf{Z} & \mathbf{V} & & & \\ & \mathbf{V}^\mathbf{T} & \mathbf{Z} & \mathbf{V} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{V}^\mathbf{T} & \mathbf{Z} & \mathbf{V} \\ & & & & \mathbf{V}^\mathbf{T} & \mathbf{Z}_m \end{bmatrix} + \begin{bmatrix} \mathbf{U}_1 & \mathbf{C}_1 & & & & \\ \mathbf{C}_1^\mathbf{T} & \mathbf{U}_2 & \mathbf{C}_2 & & & \\ & \mathbf{C}_2^\mathbf{T} & \mathbf{U}_3 & \mathbf{C}_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{C}_{m-2}^\mathbf{T} & \mathbf{U}_{m-1} & \mathbf{C}_{m-1} \\ & & & & \mathbf{C}_{m-1}^\mathbf{T} & \mathbf{U}_m \end{bmatrix},$$

where $k = \dfrac{1}{6}\dfrac{E_s}{E_c}\dfrac{A}{I} d^3 h^2$,

furthermore introducing $L_i = l_i^3$

$$\mathbf{U}_1 = \begin{bmatrix} L_2 & -2L_2 \\ -2L_2 & 4L_2 + L_3 \end{bmatrix}; \quad \mathbf{U}_m = \begin{bmatrix} L_{n-2} + 4L_{n-1} & -2L_{n-1} \\ -2L_{n-1} & L_{n-1} \end{bmatrix};$$

$$\mathbf{U}_i = \begin{bmatrix} L_{2i-2} + 4L_{2i-1} + L_{2i} & -2L_{2i-1} - 2L_{2i} \\ -2L_{2i-1} - 2L_{2i} & L_{2i-1} + 4L_{2i} + L_{2i+1} \end{bmatrix}$$

will be received, where $i = 1,2,\dots,m-1$

and $\mathbf{C}_i = \begin{bmatrix} L_{2i} & 0 \\ -2L_{2i} - 2L_{2i+1} & L_{2i+1} \end{bmatrix}$, where $i = 1,2,\dots,m-1$.

This way, performing the addition the following expressions can be received for the second order blocks of the coefficient matrix:

$$E_s A d^2 h^2 \mathbf{A} = \begin{bmatrix} \mathbf{Y}_1 & -\mathbf{B}_1 & & & & & \\ -\mathbf{B}_1^{\mathbf{T}} & \mathbf{Y}_2 & -\mathbf{B}_2 & & & & \\ & -\mathbf{B}_2^{\mathbf{T}} & \mathbf{Y}_3 & -\mathbf{B}_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\mathbf{B}_{m-3}^{\mathbf{T}} & \mathbf{Y}_{m-2} & -\mathbf{B}_{m-2} & \\ & & & & -\mathbf{B}_{m-2}^{\mathbf{T}} & \mathbf{Y}_{m-1} & -\mathbf{B}_{m-1} \\ & & & & & -\mathbf{B}_{m-1}^{\mathbf{T}} & \mathbf{Y}_m \end{bmatrix}$$

where

$$\mathbf{Y}_1 = \begin{bmatrix} L_2 + 2k & -2L_2 + k \\ -2L_2 + k & 4L_2 + L_3 + 4k \end{bmatrix},$$
$$\mathbf{Y}_m = \begin{bmatrix} L_{n-2} + 4L_{n-1} + 4k & -2L_{n-1} + k \\ -2L_{n-1} + k & L_{n-1} + 2k \end{bmatrix},$$
$$\mathbf{Y}_i = \begin{bmatrix} L_{2i-2} + 4L_{2i-1} + L_{2i} + 4k & -2L_{2i-1} - 2L_{2i} + k \\ -2L_{2i-1} - 2L_{2i} + k & L_{2i-1} + 4L_{2i} + L_{2i+1} + 4k \end{bmatrix},$$

with i = 2, 3, …, m-1 and

$$\mathbf{B}_i = \begin{bmatrix} -L_{2i} & 0 \\ 2L_{2i} + 2L_{2i+1} - k & -L_{2i+1} \end{bmatrix}.$$

The inverse matrix of $\mathbf{B}_i$ will also be needed, let us write it as follows:

$$\mathbf{B}_i^{-1} = \begin{bmatrix} -\dfrac{1}{L_{2i}} & 0 \\ \dfrac{k}{L_{2i}L_{2i+1}} - \dfrac{2}{L_{2i}} - \dfrac{2}{L_{2i+1}} & -\dfrac{1}{L_{2i-1}} \end{bmatrix}, \text{ where } i = 1,2,\dots,m-1.$$

The blocks of the inverse matrix will be expressed in the form [6]

$$\mathbf{R}_{ij} = \mathbf{P}_i \mathbf{Q}_j \text{ if } i \le j. \tag{1}$$

The matrix is symmetric, therefore

$$\mathbf{R}_{ij} = \mathbf{Q}_i^{\mathbf{T}} \mathbf{P}_j^{\mathbf{T}} \text{ if } i \ge j \tag{2}$$

The blocks **P** and **Q** can be calculated by the recursion [1]

$$
\begin{aligned}
&\mathbf{P}_1 = \mathbf{E},\\
&\mathbf{P}_2 = \mathbf{B}_1^{-1}\mathbf{Y}_1,\\
&\mathbf{P}_{i+1} = \mathbf{B}_i^{-1}\left(\mathbf{Y}_i\mathbf{P}_i - \mathbf{B}_{i-1}^{\mathbf{T}}\mathbf{P}_{i-1}\right), \quad i = 2,3,\ldots,m-1\\
&\mathbf{P}_0 = \mathbf{Y}_m\mathbf{P}_m - \mathbf{B}_{m-1}^{\mathbf{T}}\mathbf{P}_{m-1},\\
&\mathbf{Q}_m = \mathbf{P}_0^{-1},\\
&\mathbf{Q}_{m-1} = \mathbf{Q}_m\mathbf{Y}_m\mathbf{B}_{m-1}^{-1},\\
&\mathbf{Q}_i = \left(\mathbf{Q}_{i+1}\mathbf{Y}_{i+1} - \mathbf{Q}_{i+2}\mathbf{B}_{i+1}^{\mathbf{T}}\right)\mathbf{B}_i^{-1}, \quad i = m-2, m-3,\ldots,3,2,1
\end{aligned}
$$

It is to be seen that only $\mathbf{Q}_m$ needs the inversion of a single second order matrix.

5. THE LOAD VECTOR AND THE SOLUTION

Speaking about adjustment, the load vector is a vector of loading deformations, i.e. the elongation or shortening of a stay. The unit change of the length of the cable *i* produces a vertical displacement at the point *i* of the stiffening girder

$$
e_i = \frac{h}{\left\{h^2 + (i-1)^2 d^2\right\}^{\frac{1}{2}}}.
$$

The relative rotation at the hinge *i* of the primary system

$$
j_i = 2\frac{e_i}{d}
$$

and those at points *i*-1 and *i*+1 respectively

$$
j_{i-1} = j_{i+1} = -\frac{e_i}{d},
$$

because all points of the primary system will not move except point *i*. Of course, the signs can be opposite depending of lengthening or shortening of the cable-stay just being adjusted. For the sake of simplicity, let us calculate with a load vector

$$\mathbf{a}_0 = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ (2j-1 \\ (2j \\ (2j+1 \\ \\ \\ \\ \end{array} \tag{3}$$

and the result will be multiplied by a factor w depending on i, $w_i = \frac{e_i}{d}$.

To receive the unknowns, the inverse of the coefficient matrix, with blocks $\mathbf{R}_{ij}$, see (1)(2), has to be multiplied by the vector $\mathbf{a}_0$ shown in (3) and the factor w_i. Then the following expressions are obtained:

$$E_s A d^2 h^2 \mathbf{P}_i \left\{ \mathbf{Q}_j \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \mathbf{Q}_{j+1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}, \text{ if } i \le j \text{ and}$$

$$E_s A d^2 h^2 \mathbf{Q}_i^T \left\{ \mathbf{P}_j^{\mathbf{T}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \mathbf{P}_{j+1}^{\mathbf{T}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}, \text{ if } i > j,$$

where

$$\mathbf{Q}_j \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \left(\mathbf{Q}_{j+1} \mathbf{Y}_{j+1} - \mathbf{Q}_{j+2} \mathbf{B}_{j+1}^{\mathbf{T}} \right) \mathbf{B}_j^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

$$\mathbf{Q}_{j+1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \left(\mathbf{Q}_{j+2} \mathbf{Y}_{j+2} - \mathbf{Q}_{j+3} \mathbf{B}_{j+2}^{\mathbf{T}} \right) \mathbf{B}_{j+1}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{B}_j^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{L_{2j}} \\ \dfrac{2}{L_{2j}} - \dfrac{k}{L_{2j} L_{2j+1}} \end{bmatrix},$$

$$\mathbf{B}_{j+1}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{L_{2j+2}} \\ \dfrac{2}{L_{2j+2}} + \dfrac{2}{L_{2j+3}} - \dfrac{k}{L_{2j+2} L_{2j+3}} \end{bmatrix}.$$

Furthermore, since

$$\mathbf{P}_j^{\mathbf{T}}\begin{bmatrix}-1\\2\end{bmatrix}=\left(\mathbf{Y}_{j-1}\mathbf{P}_{j-1}-\mathbf{B}_{j-2}^{\mathbf{T}}\mathbf{P}_{j-2}\right)\left(\mathbf{B}_{j-1}^{\mathbf{T}}\right)^{-1}\begin{bmatrix}-1\\2\end{bmatrix},$$

$$\mathbf{P}_{j+1}^{\mathbf{T}}\begin{bmatrix}-1\\0\end{bmatrix}=\left(\mathbf{Y}_{j}\mathbf{P}_{j}-\mathbf{B}_{j-1}^{\mathbf{T}}\mathbf{P}_{j-1}\right)^{\mathbf{T}}\left(\mathbf{B}_{j}^{\mathbf{T}}\right)^{-1}\begin{bmatrix}-1\\0\end{bmatrix},$$

here

$$\left(\mathbf{B}_j^{\mathbf{T}}\right)^{-1}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}\dfrac{4k}{L_{2j-2}L_{2j-1}}-\dfrac{3}{L_{2j-2}}-\dfrac{4}{L_{2j-1}}\\-\dfrac{2}{L_{2j-1}}\end{bmatrix},$$

$$\left(\mathbf{B}_{j-1}^{\mathbf{T}}\right)^{-1}\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}\dfrac{1}{L_{2j}}\\0\end{bmatrix}$$

is to be written.

This way, the unknown moments x_j are presented and knowing these values the force S_k in an arbitrary stay k due to the adjustment applied in stay j will be as follows:

$$S_k=\frac{2x_k-x_{k-1}-x_{k+1}}{d\ h}l_k$$

6. EXAMPLE

A simple example should show the application of the method. The data of the examined cable-stayed structure are shown in Fig. 3. Two different tower heights are studied, i.e. h=20 m and h=40 m.

Let us consider that the stay 4 is shortened that way that the point moves upwards by 10 mm.

It is only possible here to plot the forces in the stays due to the above shortening for two h values in Fig. 4.

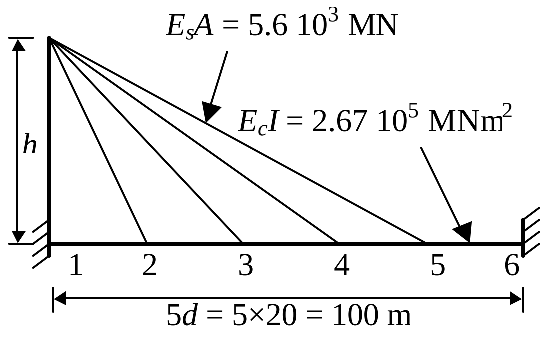

Fig. 3. Data for the example

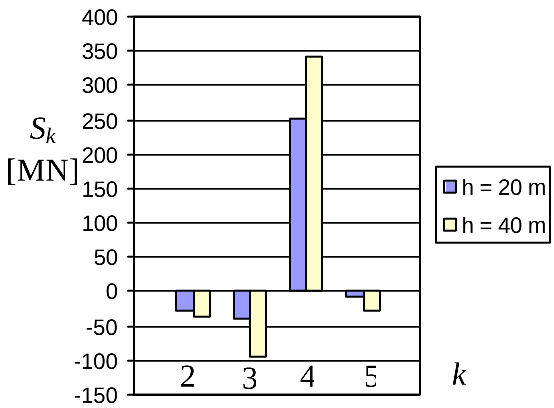

Fig. 4. Results of calculation

The method enables to carry out many other parametric analyses, of course, even for cases when there are much more cables than in this example.

7. CONCLUSION

The task was to calculate the forces in the cable-stays while the adjustment of the structure is going on, i.e. when an elongation or shortening of a single stay is performed. The problem leads to a statically indeterminate system, the degree being the number of stays (plus 2 if the ends of the stiffening girder are constrained). So, the discussion by the force method gives a linear system of equations with the above mentioned unknowns. However, the method described in this paper, using a recursion, enables to receive the solution that way that only the inverse of a single second order matrix is to be produced.

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