BASKETS

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ABSTRACT
This paper is concerned with some morphological problems of baskets with possible applications in architecture. Mainly closed baskets are considered. The old problem of bounding the sphere with hexagons is revisited. Then skewed, locally 2-way 2-fold weavings on a cube is studied. If the vertices of the net of the cube coincide with vertices of a planar square lattice, then it is ascertained that the strands form closed loops, and all loops have equal length. In such cases it is investigated: How many strands are there? The results are summarized in a table.

1. INTRODUCTION

Baskets, that people use primarily as containers in their daily life, have also architectural applications. It is well known, for instance, that woven basket skeletons have been applied as reinforcement for mud building constructions in dry areas of the world for centuries. Recently, however, new basket-like structural forms have appeared in modern architecture. A good example in small is Olafur Eliasson's pavilion in Holbaek, Denmark (Fig. 1a), and in large, Herzog & de Meuron's Beijing Olympic Stadium (Fig.1b). Both are based on quasicrystal geometry. The latter is said to be a big nest, but it can be considered equally as a giant basket where twigs are woven more or less in a free style. Another example, where not only the shape but also the material was characteristic of a basket, was the pavilion of Japan at the Aichi EXPO 2005. It was a huge real bamboo basket with the traditional three-way weaving pattern. Since appearance of more basket structures in architecture is expected in the future, it is necessary to know more about the structure and form of baskets.


(a)

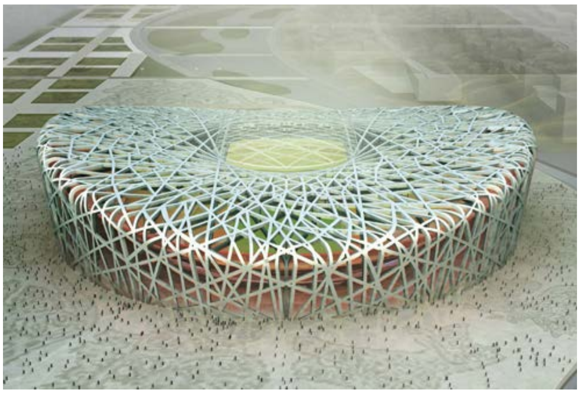
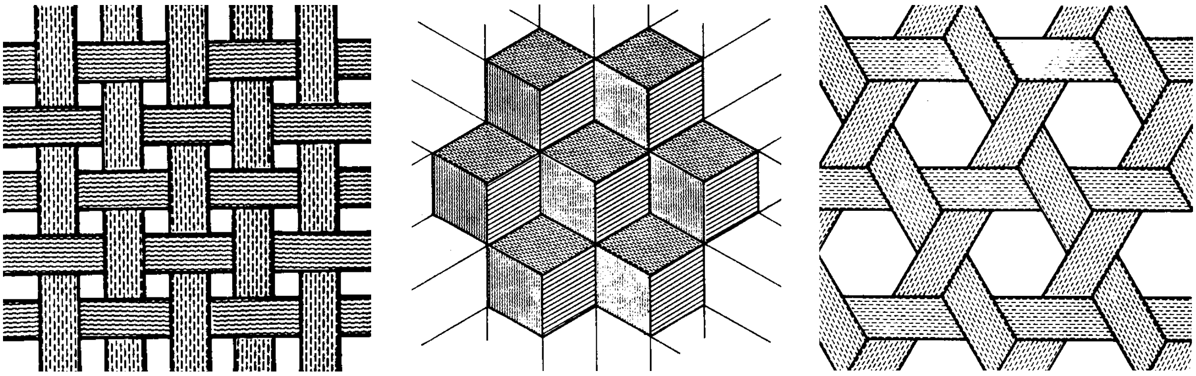

(b)

Figure 1. Basket-like structures. (a) The "5-dimensional quasi-periodic" garden pavilion by Olafur Eliasson in Holbaek, Denmark, 1998 (courtesy of Mr Eliasson). (b) The Beijing Olympic Stadium designed by Herzog & de Meuron for the 2008 Olympic Games (taken from the Internet).

In contrast with the sophisticated baskets in Fig. 1, ordinary baskets are in general much simpler, but they have a great variety of patterns and forms in weaving. The most common arrangements are where: (1) the basket is made of two layers and the strands of one layer are perpendicular to those of the other (*2-way 2-fold* weaving, Fig. 2a); (2) the basket is made of three layers and the strands of one layer incline at an angle of 60° to those of the other two (*3-way 3-fold* weaving, Fig. 2b); (3) in each of the three directions in weaving (2) every third strand is preserved and the rest is removed (*kagome*, this is the Japanese name of the obtained pattern, Fig. 2c). Each subfigure of Figure 3 shows a basket with the weaving according to the respective subfigure in Figure 2. From practical reasons actual baskets are usually made open, but there are baskets which are made closed. Figures 3(a) and (c) show closed baskets. From theoretical point of view, closed baskets are often considered as woven polyhedra or spheres [1]. Mathematical aspects of weaving patterns and shapes of baskets including that of closed baskets have been studied e.g. by Pedersen [1], Grünbaum and Shephard [2], Gerdes [3], von Randow [4]. Significant contribution to the basket research has been made by Iijima [5] who discovered carbon nanotubes and used baskets in his studies. He has delivered several lectures on baskets and their connection to fullerenes and carbon nanotubes but, unfortunately, has not published anything about his results on baskets in written form.



(a) (b) (c)

Figure 2. Types of weaving: (a) 2-way 2-fold, (b) 3-way 3-fold, (c) kagome (after J.J. Pedersen, B. Grünbaum and J.C. Shephard).

(a) (b) (c)

Figure 3. Baskets made in weave shown in the respective subfigures of Fig.2. (a) Closed tetrahedron. Ration basket for grain, millet, Somalia (Pitt Rivers Museum, Oxford). (b) Basket of palm-leaf, Isle of Crete. (c) Closed bamboo basket for keeping a cricket, China.

The purpose of this paper is to investigate baskets from the point of view of structural morphology. The actual aim is twofold. Firstly, to show how curvature is made for kagome baskets, and to draw attention to an old but often neglected result on bounding a sphere with hexagons. Secondly, to discover the main geometrical properties of skewed, locally 2-way 2-fold weavings on the surface of a cube. Since for these cases it will be ascertained that the strands form closed loops, and all loops have equal length, the main question is: How many strands are there?

2. MAKING CURVATURE

Consider the kagome weave in Fig. 2(c). The geometrical model of its pattern is the Archimedean plane tessellation denoted by the Schläfli symbol (3,6,3,6), that means that every vertex of the tessellation is surrounded by a triangle, a hexagon, a triangle and a hexagon in this cyclic order [6]. Centres of triangles determine the {6,3} regular plane tessellation composed of equal hexagons, three around a vertex [6]. Therefore, there is a one-to-one correspondence between the kagome pattern and the regular hexagon tessellation.

Let us select a hexagon of a kagome pattern drawn on a sheet of paper, take its centre point, and from that point draw arbitrarily two half lines enclosing an angle of $\pi/3$. Let us cut out and remove this $\pi/3$ sector from the kagome pattern, then bring the two half lines into coincidence. In this way a cone is obtained where, because of the sixfold rotational symmetry of kagome, all lines of the pattern go through the cutting half line smoothly. With this operation, the hexagon becomes a pentagon at the vertex of the cone. Discrete curvature at a point (now at the vertex of the cone) is defined in a way that it is proportional to the angular deficiency (now $\pi/3$). Lines, which were straight in the plane, now are geodesics on the surface of the cone. Therefore, the kagome pattern on the cone can be realized by weaving of strands. Indeed, this is the way, how Southeast Asian bamboo hats are made (Fig. 4a). By repeating the above argument, additional pentagons can be introduced into the weaving system. Appearance of every pentagon increases the aggregate angular deficiency by $\pi/3$. Figure 4 shows kagome baskets with 1 to 6 pentagons. Figure 4 also shows that with increasing the number of pentagons, the basket form gradually changes from a cone to a singly capped cylinder.

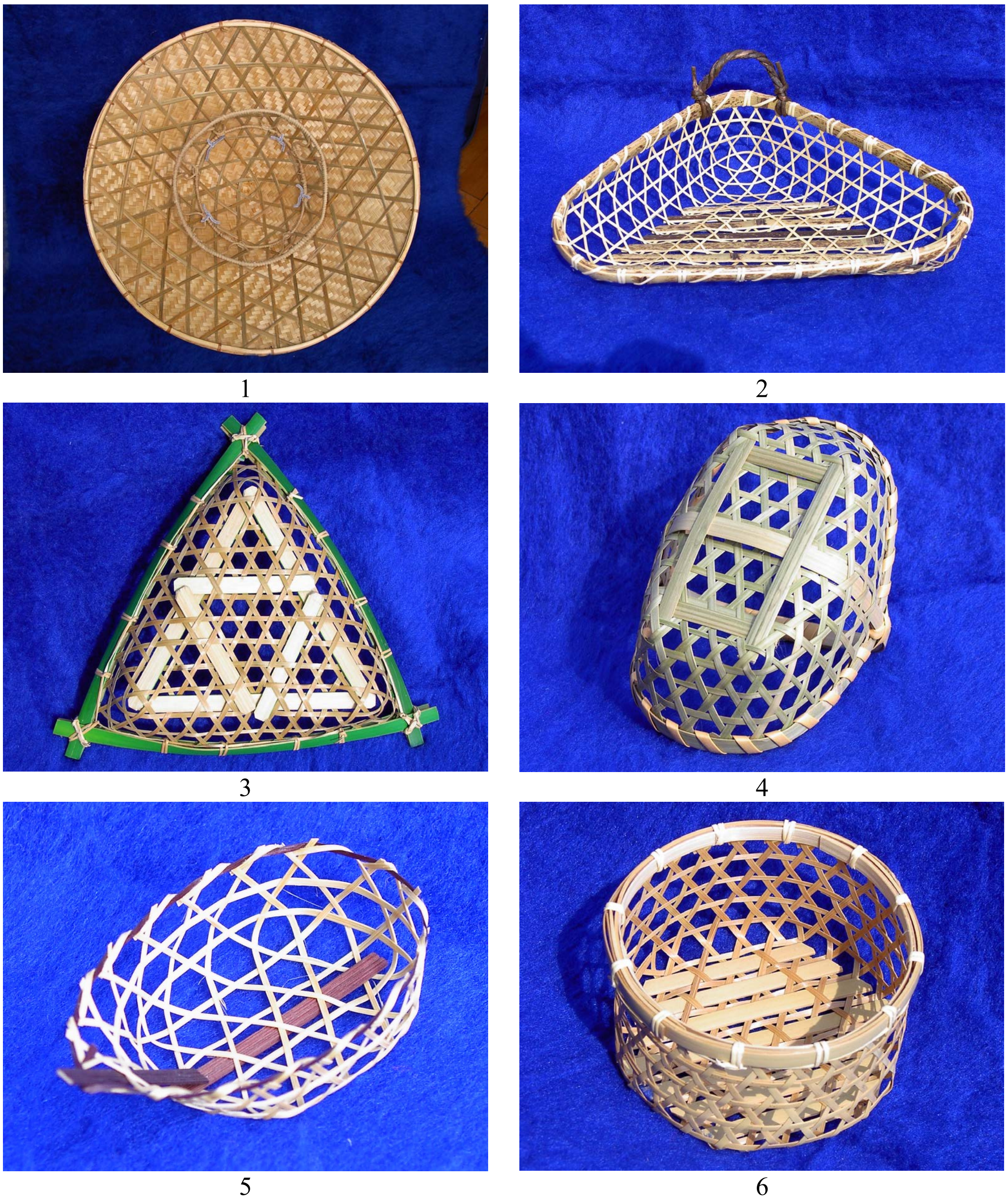
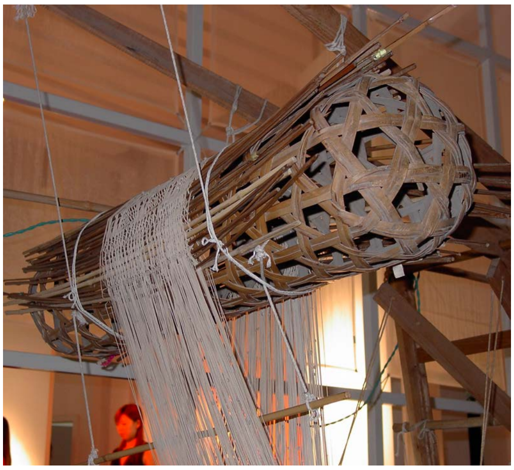


Figure 4. Bamboo baskets with 1 to 6 pentagons in the kagome weaving pattern, Kyoto, Japan.

The cylindrical form for 6 pentagons is not an accident. In the case of 6 pentagons, the aggregate angular deficiency is 2π. According to Descartes' theorem on polyhedra [7], the total angular deficiency of a polyhedron topologically equivalent to a sphere is 4π. It follows that a kagome basket with 6 pentagons corresponds to a "hemisphere" that can be extended with a cylinder. The woven cylindrical surface can be arbitrarily long for any strand orientation. Another important corollary of the Descartes theorem is that, for a polyhedron bounded by hexagons and pentagons, where at each vertex 3 edges meet, *the number of pentagons is 12*. Indeed, if we took two copies of the basket in Fig. 4.6 and put one on top of the other up side down, we would obtain a closed basket of 12 pentagons that would be similar to the basket in Fig. 3(c). This corollary, that can be considered as a theorem, is topological (combinatorial) in nature and is independent of the arrangement of the pentagons. Therefore, it is equally valid for the case where the pentagons are arranged at the two ends of a bicapped tube (Fig. 5a) and for the case where they are distributed uniformly on a sphere and put at the vertices of an icosahedron (Fig. 5b).

The above theorem is known also in the form: *No system of hexagons can enclose a simply connected domain in space. This is possible, if also pentagons are applied and each vertex is surrounded by three polygons, then the number of pentagons is always 12 independently of the number of hexagons.* This theorem can be proved easily by using Euler's polyhedron theorem. The proof can be found in several textbooks and monographs such as D'Arcy Thompson's book [8]. Interestingly, basket-makers know this theorem from practice, but many architects, who ought to know this, do not know.

(a) (b)

Figure 5. Closed kagome bamboo baskets with 12 pentagons. (a) Tubular. Part of a silk weaving machine, Silk Museum, Hangzhou, China. (b) Spherical (icosahedral). Skeleton of flower decoration for a Bon festival, Kyoto, Japan (courtesy of Prof. K. Miyazaki).

(a) (b)

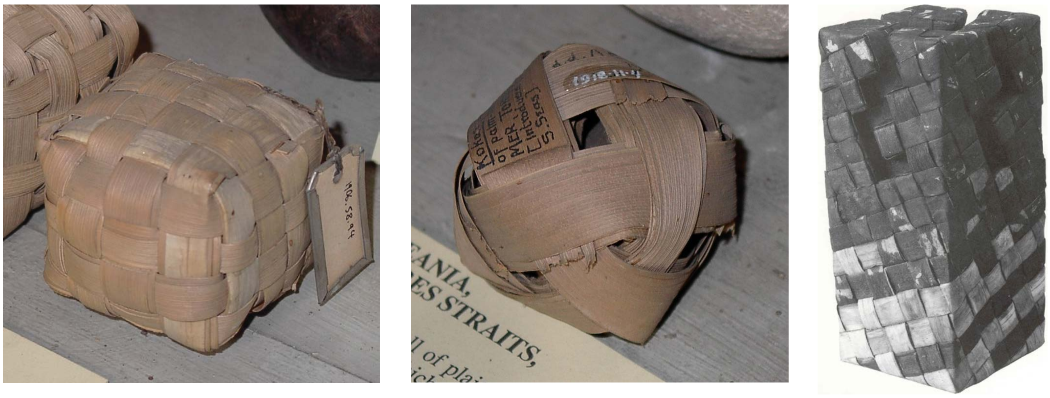
Figure 6. Kagome bamboo baskets with 6 pentagons and 6 pentagon-heptagon pairs, Kyoto, Japan. Arrangements having (a) C_{6v} symmetry and (b) C_{2v} symmetry.

Heptagons can also occur in kagome baskets. According to the above argument, introduction of a heptagon causes a $-\pi/3$ angular deficiency ($\pi/3$ angular excess) in the kagome system. As mentioned before, introduction of a pentagon causes a $+\pi/3$ angular deficiency. Therefore, a simultaneous introduction of a heptagon and a pentagon does not cause any change in the aggregate angular deficiency. Consequently, in a closed kagome basket, the *difference between the number of pentagons and the number of heptagons is always 12*. (This result comes also from the above-mentioned theorem based on Euler's polyhedron theorem.) This is why always a pentagon is associated to a heptagon. They occur together, usually side by side as seen in Fig. 6 which shows two different arrangements of 6 pentagon-heptagon pairs in the kagome system corresponding to a "hemisphere".

3. WEAVING ON THE SURFACE OF THE CUBE

3.1. Basic properties

In this paper only such planar weaves are studied where the strands have equal width and the weaving pattern is lattice-like (the whole pattern can be generated by translation of the unit cell of the pattern). These requirements are fulfilled, for instance, for the basic weaving forms in Fig. 2. A 2-way 2-fold weave produces a check pattern that is easy to execute with the strands: repeatedly 1 under, 1 above, and the translation symmetry of the pattern is maintained. In the space, however, if we want to use this weaving system on the surface of a cube, the translation symmetry is broken at the vertices of the cube, and the strands run in more than 2 directions. The global weaving system of the plaited cube in Fig. 7(a) is 3-way 2-fold, and that in Fig. 7(b) is 4-way 2-fold (*c.f.* Ref [1]), while on the faces of the cube, being planar, the weaving system locally is 2-way 2-fold. These two weaving systems, where the strands run parallel to the edges and where the strands incline to the edges at an angle of $\pi/4$, have been used for baskets with rectangular (square) base for ages. Not long ago, however, it was shown that it is possible to make a basket also in a skewed weave, with strand inclination at an angle different from 0 ($\pi/2$) or $\pi/4$ (Fig. 7c).



(a) (b) (c)

Figure 7. Locally 2-way 2-fold weaving. (a) A cube of plaited pandanus leaf with strands parallel to the edges, Oceania, Tuvalu (Ellice Ids.). (b) A cube of plaited palm-leaf with strands inclining at an angle of $\pi/4$ to the edges, Oceania, Torres Straits, Mer. (Both cubes are in the Pitt Rivers Museum, Oxford). (c) Skewed weaving on a square based prism by the Canadian artist John McQueen, 1977 (courtesy of Ms Felicity Wood).

It is obvious that, in skewed weaving on a cube, the angle of inclination of a strand cannot be arbitrary. The reason for that is that a strand cannot contain a vertex of the cube as an internal point. A vertex of the cube can only be situated on the boundary of a strand. This property provides a discrete character for the angle of inclination. Skewed weavings can be extended to the complete cube, and the different possibilities can be investigated in the following way. Consider the regular tessellation of Coxeter symbol $\{4,3+\}_{b,c}$ which consists of squares, three or more (four) at a vertex, some slightly folded, such that they cover and fill the surface of the cube, in which a vertex of the cube can be arrived at from an adjacent one along the edges of the tessellation by b steps on the vertices of the tessellation in one direction then c steps after a change in direction by an angle of $\pi/2$ (Fig. 8). The non-negative integers b and c such that $b^2 + c^2 \neq 0$ determine the square number S, $S = b^2 + c^2$, that provides the number of small squares on a face of the cube, or the area of the face of the cube if the area of a small square is the unity [9]. If a square of the tessellation is translated along the edges of the tessellation in one direction, then the trace of the translated square determines a strip that can be considered as a strand. With this notation, the underlying tessellation for the weave in Fig. 7(a) is $\{4,3+\}_{4,0}$, for that in Fig. 7(b) is $\{4,3+\}_{1,1}$, and after proper truncation of the prism in Fig. 7(c) it is $\{4,3+\}_{2,5}$.

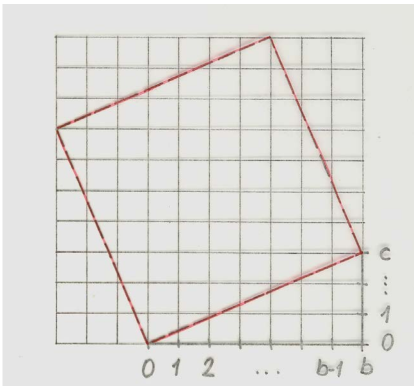


Figure 8. Definition of parameters b and c. The rotated large square is a face of the cube.

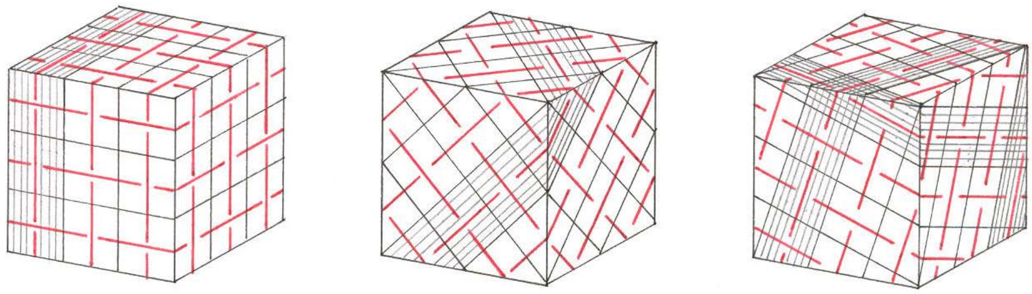
According to the relation between b and c, square tessellations on the surface of the cube form three classes.

(a) $b = 0$ or $c = 0$, $b \neq c$,

(b) $b = c$, $b \neq 0$,

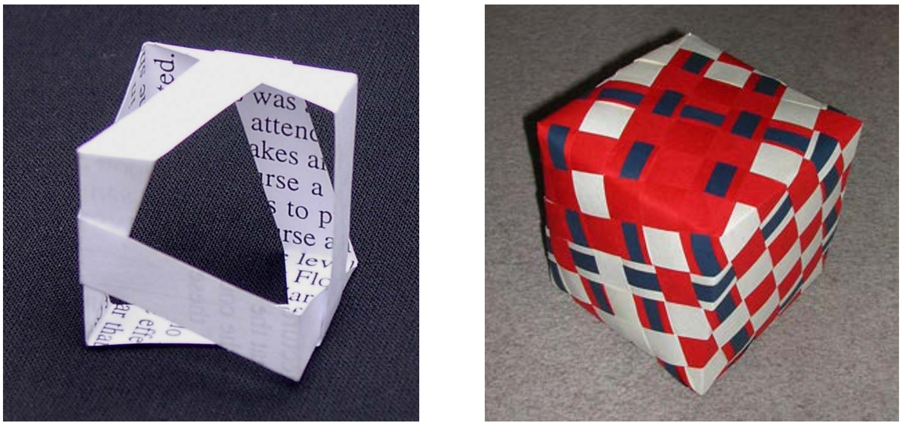
(c) $b \neq c$, $b \neq 0$, $c \neq 0$.

Tessellations in (a) and (b) have planes of symmetry, but in (c) have no plane of symmetry. Locally 2-way 2-fold weaves on a cube can be classified according to the classification of the underlying square tessellations. In class (a), strands run parallel (or perpendicular) to the edges of the cube; in class (b), strands incline to the edges of the cube at an angle of $\pi/4$ while in class (c), strands incline to the edges of the cube at an angle different from 0 ($\pi/2$) or $\pi/4$ (Fig. 9). (The underlying tessellations of the actual weaves in Fig. 9 are: (a) $\{4,3+\}_{0,3}$, (b) $\{4,3+\}_{2,2}$, (c) $\{4,3+\}_{3,1}$.) Weaves in Fig. 7(a), (b) and (c) belong to weaving class (a), (b) and (c), respectively.



(a) (b) (c)

Figure 9. Classes of locally 2-way 2-fold weavings on a cube. (a) Strands are parallel to the edges. (b) Strands incline to the edges at an angle of $\pi/4$. (c) Skewed weaving.



(a) (b)

Figure 10. One loop of a strand for (a) $b = 3$, $c = 1$; (b) $b = 2$, $c = 5$, the dark strand shows one loop (courtesy of Ms Felicity Wood).

Without detailed proof we give a short overview on the main properties of stands. The midline of a strand is straight in the net of the cube in the plane. Therefore, it is a geodesic on the cube. Parameters b and c are non-negative integers, so c/b is a rational number, consequently, the midlines form closed geodesics, and the strands form closed loops. Figure 10 shows examples of loops for different values of b and c. If b and c are relative primes, that is, b and c have no divisor in common other than 1, then all loops are congruent. This fact comes from the O symmetry of the woven cube and the actual symmetry of the loops. For any given pair b, c, the lengths of all loops are equal. This is yielded from the fact that, if k is the greatest common divisor of b and c, then they have the form $b = kb_1$ and $c = k\,c_1$ where b_1 and c_1 are relative primes, and for b_1 and c_1 we have congruent loops with k times greater width. Cutting such a loop into k narrow loops, the obtained loops will have the same length as the original has. If we just wind a strand on the cube without weaving, then take the cube out of the obtained loop, and after cutting through the loop we open it, we find very often that the strand is twisted. The number of twists (full turns) denoted by t is in connection with the winding number w: $t = w - 1$. Consider a weave on a cube and remove all loops except one. If we take the cube out of the remaining loop, we find very often that the loop forms a knot. This can be seen also in Fig. 10a.

3.2. The main questions

After knowing these properties we are looking for answers to the following questions for given values of b and c for a woven cube.

(i) How many strands are there?
(ii) How large twist does a strand have? (What is the value of the number t of full turns?)
(iii) What sort of knot does a strand have?
(iv) If b and c are relative primes, then what sort of symmetry does a strand have?

3.3. Results

We can answer the four questions experimentally using physical models. Taking a solid cube and a long tape of paper, we can wind the tape on the cube, in correspondence with the actual values of b and c, until the tape is closed. Then we measure the length of the closed tape and calculate the area of one strand. Because the strands cover the surface of the cube exactly twice, the number of strands, let us denote it by n, is easily obtained. After straightening the tape we can count the full twists. Knot type and symmetry can be determined on inspection.

Table 1. Number n of the closed strands (loops) for different values of b and c.

c b	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57
1	3	4	6	4	3	4	6	4	3	4	6	4	3	4	6	4	3	4	6	4
2	6	6	8	3	12	3	8	6	6	6	8	3	12	3	8	6	6	6	8	3
3	9	4	3	12	3	4	18	4	3	12	3	4	9	4	3	12	3	4	18	4
4	12	3	12	3	16	6	6	3	24	3	6	6	16	3	12	3	12	3	12	3
5	15	4	3	4	6	20	3	4	3	4	30	4	3	4	3	20	6	4	3	4
6	18	6	8	18	6	3	24	3	6	9	8	3	36	3	8	9	6	3	24	3
7	21	4	6	4	3	4	3	28	6	4	6	4	3	4	42	4	3	4	6	4
8	24	3	6	3	24	3	6	6	32	3	12	6	12	6	6	3	48	3	6	6
9	27	4	6	12	3	4	9	4	3	36	3	4	9	4	6	12	3	4	54	4
10	30	6	8	3	6	30	8	6	12	3	40	6	6	3	8	15	6	6	8	3
11	33	4	3	4	6	4	3	4	6	4	6	44	3	4	3	4	3	4	3	4
12	36	3	12	9	16	3	36	3	12	9	6	3	48	3	6	18	12	3	18	6
13	39	4	3	4	3	4	3	4	6	4	3	4	3	52	6	4	3	4	3	4
14	42	6	8	3	12	3	8	42	6	6	8	3	6	6	56	3	12	3	8	3
15	45	4	6	12	3	20	9	4	3	12	15	4	18	4	3	60	3	4	9	4
16	48	3	6	3	12	6	6	3	48	3	6	3	12	3	12	3	64	6	6	6
17	51	4	6	4	3	4	3	4	3	4	6	4	3	4	3	4	6	68	3	4
18	54	6	8	18	12	3	24	6	6	54	8	3	18	3	8	9	6	3	72	3
19	57	4	3	4	3	4	3	4	6	4	3	4	6	4	3	4	6	4	3	76

Applying this method for $b = 3$ and $c = 1$, we found that $n = 4$, $t = 1$, the strand forms a trefoil knot [10] as shown in Fig. 10(a), and its symmetry is D_3. Later we focused on the question (i) only. It turns out that the answer to (i) for $c = 1$, $b = 1,2, \dots$, that is, for woven cube based on the tessellation $\{4,3+\}_{b,1}$ was given by Pedersen and Shephard [11]. My colleague G. Károlyi worked out a computer program by which it is possible to determine the number n of the closed strands (loops) for any values of b and c. The results of the calculations are summarized in Table 1, and in a more visual form in Fig. 11. The data in Table 1 show some interesting features. The Table as a matrix is symmetric (emphasized by a secondary counter-diagonal highlighted with green), expressing the fact that interchanging b and c changes only the chirality of the weave but not the number of strands. For given c, the sequence of n's is periodic with a period of $4c$, and the ith period is symmetric with respect to the point of the line $c = (4i - 2)^{-1}b$ at the given c. Consider, for example, the first period ($i = 1$) for $c = 2$ (highlighted with yellow) that is symmetric with respect to $b = 4$ (that is number 12 in blue; other blue numbers in the Table represent centres of symmetry of the first periods). Because of periodicity, if $b \equiv b_1 \pmod{4c}$, then $n(b,c) = n(b_1,c)$, that is, the same number of strands appear in different geometries (on the same cube with the same strand surface area). If b and c are relative primes, then $n = 3, 4, 6$. If $b = kb_1$, $c = kc_1$, such that b_1, c_1 are relative primes and $k > 0$ integer, then $n(b,c) = kn(b_1,c_1)$. This is in agreement with the above-mentioned properties of closed strands. This property at the same time calls attention to the fact that the really important cases are where b and c are relative primes.

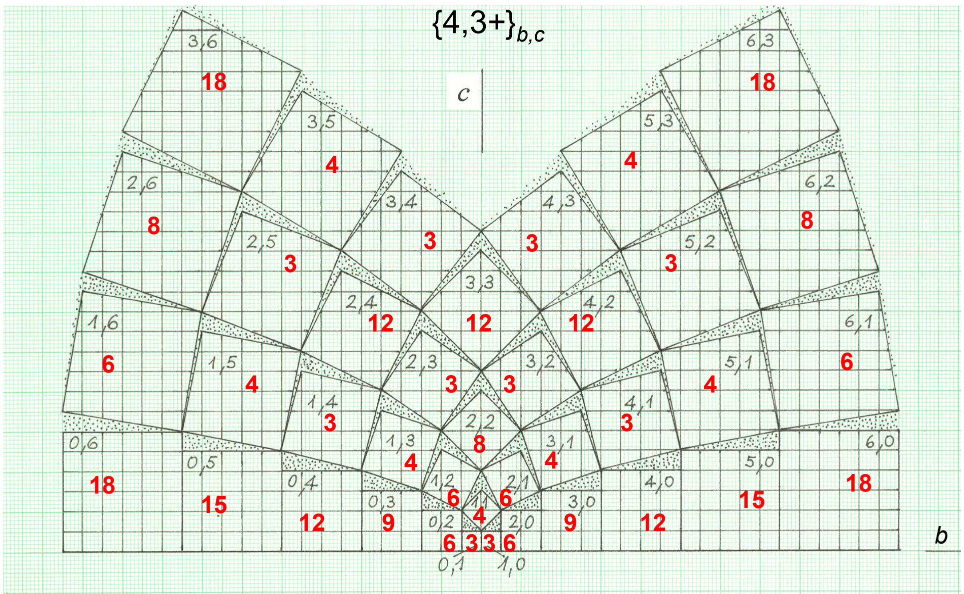


Figure 11. Number n of the closed strands (loops) for different values of b and c. The large squares represent a face of the cube with the actual part of the square tessellation on it. The figures show symmetry if b and c are exchanged.

4. CONCLUDING REMARKS

As mentioned, baskets provide new forms in contemporary architecture, and with appropriate selection of structural material, they can appear as environmentally friendly space frames. A good example of that was the Japanese basket pavilion at the Aichi EXPO 2005, which was made of bamboo. Experts, however, discovered that, at this pavilion, the kagome pattern formed by the supporting bamboo bars did not contain pentagons, though the bars were fitted to a double-curved surface. In this paper, we drew attention to some elementary rules in the topology of baskets (e.g. 12 pentagons are needed in a closed kagome cage), that can help architects and structural engineers to design the geometry of basket structures more professionally in the future.

Locally 2-way 2-fold weaves on the surface of a complete cube show a number of surprising mathematical properties. One of them is: if the b, c parameters of weaving are relative primes, then the number of closed strands can be only 3, 4, 6. These numbers are certainly in connection with the values of b and c and with the octahedral symmetry group O (and its subgroups) of the woven cube, but the *how* is still an open question. Another unanswered question is: Why do the numbers n of the closed strands form a periodic sequence for given c, and why are the numbers inside a period arranged symmetrically?

In this paper, for a woven cube, we studied only the number of closed strands as a function of b and c: $n(b,c)$. The analysis of the number of full twists, $t(b,c)$, the symmetry and the knot type of a closed strand is left for future research, that can be extended to the topological investigation of the closed basket as a whole using the theory of knots and links [12].

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